

Soliton solutions for Q3

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Abstract. We construct N-soliton solutions to the equation called Q3 in the recent Adler-Bobenko-Suris classification. An essential ingredient in the construction is the relationship of $(Q3)_{\delta=0}$ to the equation proposed by Nijhoff, Quispel and Capel in 1983 (the NQC equation). This latter equation has two extra parameters, and depending on their sign choices we get a 4-to-1 relationship from NQC to $(Q3)_{\delta=0}$. This leads to a four-term background solution, and then to a 1-soliton solution using a Bäcklund transformation. Using the 1SS as a guide allows us to get the N-soliton solution in terms of the tau-function of the Hirota-Miwa equation.

1. Introduction

Integrable lattice equations have a long history, going back to pioneering work in the 1970's, [1, 2], with subsequent development of systematic approaches in the early 1980's, [3, 4, 5], cf. also the review [6]. It has long been known, in fact it is implicit in these constructions, that the emerging examples possess the property of “multidimensional consistency”. By this we mean the property that the partial difference equation describing the lattice systems can be consistently embedded in a multidimensional space, with in principle an infinity of lattice variables. Most succinctly this property was described in recent years in [7], in a context where it was explicitly used to achieve multi-dimensional reductions, and it was subsequently re-appraised in [8]. This multidimensional consistency is a very natural property, it is the precise analogue of the well known existence of compatible higher time-flows in hierarchies of soliton systems, and as such it was quite well understood in the earlier publications mentioned.

In [9] the property of “consistency around a cube” (CAC) was used to classify partial difference equations defined on an elementary square of a two-dimensional lattice. Remarkably, within certain restrictive conditions (symmetry and the “tetrahedron condition”), a full list of scalar quadrilateral lattice equations could be established, and this list is surprisingly short. (In a more recent paper [10] the classification statement was proven under slightly less restrictive conditions.)

The list of CAC-integrable lattice equations in [9] contains some interesting examples, but the equation at the top of the list (denoted as Q4 in [9]) was actually found earlier by V. Adler, cf. [11]. This equation, which we refer to as *Adler's equation*, is in fact an integrable discretization of the famous Krichever-Novikov (KN) equation,

[12, 13]. For Adler's equation a Lax pair was established in [14] on the basis of its multidimensional consistency, and in [15] this equation was shown to be a master equation among various well-known integrable systems associated with an elliptic curve. The first solutions for Adler's equation were established in a recent paper [16], cf. also [17] for a slight generalisation of those results. However, so far little is known about the algebraic structure behind Adler's equation, and since this equation has lattice parameters which lie on an elliptic curve, the underlying structure is expected to be interesting but rather complicated. Thus, before tackling the Adler's equation, it is of interest to study some of the other examples emerging from the list of [9], in order to see how the underlying structure of such equations can be revealed.

Here we focus on the equation denoted as Q3 in [9] and which is just below Q4 in the hierarchy. It is written as:

$$\begin{aligned} \mathring{p}(1 - \mathring{q}^2)(u\widehat{u} + \widetilde{u}\widehat{\widetilde{u}}) - \mathring{q}(1 - \mathring{p}^2)(u\widetilde{u} + \widehat{u}\widehat{\widetilde{u}}) - (\mathring{p}^2 - \mathring{q}^2)(\widehat{u}\widetilde{u} + u\widehat{\widetilde{u}}) \\ = \delta^2(\mathring{p}^2 - \mathring{q}^2)(1 - \mathring{p}^2)(1 - \mathring{q}^2)/(4\mathring{p}\mathring{q}). \end{aligned} \quad (1.1)$$

In (1.1) we have adopted the convention of representing shifts in the rectangular lattice with tildes and hats: The corners of an elementary plaquette on a rectangular lattice are thus

$$u \equiv u_{n,m}, \quad \widetilde{u} \equiv u_{n+1,m}, \quad \widehat{u} \equiv u_{n,m+1}, \quad \widehat{\widetilde{u}} \equiv u_{n+1,m+1}.$$

The *lattice parameters* \mathring{p} , \mathring{q} in (1.1) are associated with the n , m directions in the lattice, respectively, (they can be thought of as measuring the grid size in these directions) while δ is a global parameter.

In this paper we construct a general family of N -soliton solutions of Q3. The construction of these solutions is based on the relationship of (1.1) to a lattice equation that first appeared in [3] (cf. also [4]) about 25 years ago, and which is equivalent to the $\delta = 0$ case of Q3. The explicit 4-to-1 relationship between these two equations is explained in Section 2, and it requires the introduction of another parametrisation which is more suitable for the solution. The N -soliton solution of Q3 with parameter δ (denoted by $(Q3)_\delta$) then appears as a *linear* combination of four independent solutions of the $\delta = 0$ case, $(Q3)_0$, with four arbitrary coefficients subject to one single relation. We have obtained this result in two different ways, one of which employs a novel Miura transformation (explained in the Appendix), which allows us to derive soliton solutions for $(Q3)_\delta$ from the known solutions of $(Q3)_0$ of [3, 4]. We will not present this approach here, because it requires quite a bit of a notational machinery for which we lack space in this note, and this approach will be published elsewhere, [18].

Alternatively, we can use the 4-to-1 correspondence which appears through the introduction of some additional lattice shifts associated with the parametrisation given in Section 2, to obtain a general “seed” solution for the relevant Bäcklund transformations. This allows us in Section 3 to obtain the 1-soliton solution in a form which suggests a τ -function description. We will present the N -soliton solution of Q3 in Hirota form in Sec. 4, and show how the solution is related to a set of discrete Hirota-Miwa equations in a four-dimensional lattice.

2. The basic lattice equations

The special case $\delta = 0$ of (1.1), appeared for first time in [3], (cf. also [4]) in the form

$$\frac{1 + (p - a)s - (p + b)\tilde{s}}{1 + (q - a)s - (q + b)\hat{s}} = \frac{1 + (q - b)\tilde{s} - (q + a)\hat{\tilde{s}}}{1 + (p - b)\hat{s} - (p + a)\hat{\tilde{s}}}. \quad (2.1)$$

We call this the NQC equation, following [19].

To bring the equation (2.1) to the form of $(Q3)_0$ we perform the transformation ($a + b \neq 0$):

$$u_{n,m} = \tau^n \sigma^m \left(s_{n,m} - \frac{1}{a+b} \right) \quad , \quad (2.2)$$

where

$$\tau := \sqrt{\frac{(p+a)(p+b)}{(p-a)(p-b)}} \quad , \quad \sigma := \sqrt{\frac{(q+a)(q+b)}{(q-a)(q-b)}}. \quad (2.3)$$

This yields $(Q3)_0$ with the parametrization

$$P(u\hat{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - Q(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - (p^2 - q^2)(\hat{\tilde{u}}\tilde{u} + u\hat{\tilde{u}}) = 0, \quad (2.4)$$

where the lattice parameters have now become points $\mathfrak{p} = (p, P)$ and $\mathfrak{q} = (q, Q)$, respectively, on the (Jacobi) elliptic curve:

$$P^2 = (p^2 - a^2)(p^2 - b^2) \quad , \quad Q^2 = (q^2 - a^2)(q^2 - b^2). \quad (2.5)$$

In this parametrization the $\delta \neq 0$ version of (1.1) reads

$$P(u\hat{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - Q(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - (p^2 - q^2)(\hat{\tilde{u}}\tilde{u} + u\hat{\tilde{u}}) = \delta^2 \frac{(p^2 - q^2)}{4PQ}. \quad (2.6)$$

Note that the original parametrization of Q3 in (1.1) is subtly different from (2.4) and they can be related by the identifications:

$$\hat{p}^2 = \frac{p^2 - b^2}{p^2 - a^2}, \quad \hat{q}^2 = \frac{q^2 - b^2}{q^2 - a^2}, \quad P = \frac{(b^2 - a^2)\hat{p}}{1 - \hat{p}^2}, \quad Q = \frac{(b^2 - a^2)\hat{q}}{1 - \hat{q}^2}. \quad (2.7)$$

In fact, the two different parametrizations presented in (1.1) and (2.6) are just different ways to parametrize the equation $\alpha(u\hat{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - \beta(u\tilde{u} + \hat{\tilde{u}}\hat{\tilde{u}}) - \gamma(\hat{\tilde{u}}\tilde{u} + u\hat{\tilde{u}}) = \delta$, with the constraints arising from CAC (related to dependency on lattice parameters associated with different directions of the cube).

It is important for later to observe that the parametrizations of P, Q, \hat{p}, \hat{q} are invariant under the sign change of a and/or b while the NQC equation itself is not. This means that there are in fact *four different versions* of NQC (corresponding to these sign changes) and they all provide a different solution to $(Q3)_0$, though the transformation (2.2) with corresponding sign changes in it and (2.3). We will use this 4-to-1 correspondence later to construct solutions to Q3.

It was shown in [3] that (2.1) interpolates through different choices of the auxiliary parameters a, b between various lattice equations “of KdV type”, and hence could be thought of as an interpolating equation. We can identify, e.g., the following subcases of (2.1) appearing in the list of [9] (up to gauge transformations, where necessary):

$a = b = 0 \Rightarrow (Q_1)_0$, $a = 0, b \rightarrow \infty \Rightarrow (H_3)_0$ and $a, b \rightarrow \infty \Rightarrow H_1$, which are respectively the lattice Schwarzian KdV equation, the lattice potential modified KdV equation and the lattice potential KdV equation. For all these subcases N -soliton solutions can be given in closed form, which follows as an immediate application of the direct linearisation scheme elaborated in [3, 4].

3. Background and one-soliton solutions

3.1. Seed or background solution for Q3

The trivial solution to (2.1) is $s_{n,m} \equiv 0$ and from this it follows that $u_{n,m} = c\tau^n\sigma^m$ is a background or “seed” solution for $(Q3)_0$. Furthermore, as discussed before, by changing the signs of a and/or b we get other seed solutions for $(Q3)_0$, namely

$$u_{n,m}^{++} = A\tau^n\sigma^m, \quad u_{n,m}^{--} = B\tau^{-n}\sigma^{-m}, \quad u_{n,m}^{+-} = C\dot{\tau}^n\dot{\sigma}^m, \quad u_{n,m}^{-+} = D\dot{\tau}^{-n}\dot{\sigma}^{-m}, \quad (3.1)$$

where τ and σ were defined in (2.3) and

$$\dot{\tau} := \sqrt{\frac{(p+a)(p-b)}{(p-a)(p+b)}}, \quad \dot{\sigma} := \sqrt{\frac{(q+a)(q-b)}{(q-a)(q+b)}}. \quad (3.2)$$

Starting with one such seed solution for $(Q3)_0$ one can use a Miura transformation (see Appendix) to derive a solution for $(Q3)_\delta$. The result turns out to be a linear combination of three of the above terms, and leads us to try a linear combination of all four terms, that is

$$u_{n,m}^{0SS} = A\tau^n\sigma^m + B\tau^{-n}\sigma^{-m} + C\dot{\tau}^n\dot{\sigma}^m + D\dot{\tau}^{-n}\dot{\sigma}^{-m}. \quad (3.3)$$

It is easy to verify that this is indeed a solution of $(Q3)_0$ provided that

$$AB(a+b)^2 - CD(a-b)^2 = 0, \quad (3.4)$$

and, perhaps surprisingly, that (3.3) is also a solution of $(Q3)_\delta$, provided that

$$AB(a+b)^2 - CD(a-b)^2 = -\frac{\delta^2}{16ab}. \quad (3.5)$$

From this last result we see, that when $\delta \neq 0$ one cannot use any *single* seed given in (3.1) as a starting solution, and for a “germinating seed” (in the terminology of [16]) one needs at least the pair $++$, $--$ or the pair $+-$, $-+$.

3.2. 1-Soliton solution $(Q3)_0$ from BT

Having obtained a nontrivial background solution (3.3), we can now proceed to construct soliton solutions starting from this seed solution of the Bäcklund transformation (BT). From cubic consistency it follows that we can impose in addition to the lattice equation (2.4) also the set of equations:

$$P(u\bar{u} + \widetilde{u}\widetilde{\bar{u}}) - K(u\widetilde{u} + \bar{u}\widetilde{\bar{u}}) = (p^2 - k^2)(\widetilde{u}\bar{u} + u\widetilde{\bar{u}} + \frac{\delta^2}{4PK}), \quad (3.6a)$$

$$Q(u\bar{u} + \widehat{u}\widehat{\bar{u}}) - K(u\widehat{u} + \bar{u}\widehat{\bar{u}}) = (q^2 - k^2)(\widehat{u}\bar{u} + u\widehat{\bar{u}} + \frac{\delta^2}{4QK}), \quad (3.6b)$$

where $K^2 = (k^2 - a^2)(k^2 - b^2)$. Since the “bar”-direction is for increasing number of solitons, we search for a new $\bar{u}(\equiv u^{1SS})$ of the form: $\bar{u} = \bar{u}_\theta + v$, where \bar{u}_θ is the bar-shifted background solution (3.3):

$$\bar{u}_\theta = A\tau^n \sigma^m \kappa + B\tau^{-n} \sigma^{-m} \kappa^{-1} + C\dot{\tau}^n \dot{\sigma}^m \dot{\kappa} + D\dot{\tau}^{-n} \dot{\sigma}^{-m} \dot{\kappa}^{-1}, \quad (3.7)$$

where

$$\kappa = \sqrt{\frac{(k+a)(k+b)}{(k-a)(k-b)}}, \quad \dot{\kappa} = \sqrt{\frac{(k+a)(k-b)}{(k-a)(k+b)}}. \quad (3.8)$$

(Note however, that we can absorb the $\kappa, \dot{\kappa}$ factors into A, B, C, D without changing the relation (3.4) or (3.5).)

From (3.6) one can then solve

$$\tilde{v} = \frac{[(p^2 - k^2)\tilde{u}_\theta + K\tilde{\bar{u}}_\theta] - Pu_\theta}{-Kv + [P\tilde{u}_\theta - K\bar{u}_\theta - (p^2 - k^2)u_\theta]}, \quad \hat{v} = \frac{[(q^2 - k^2)\hat{u}_\theta + K\hat{\bar{u}}_\theta] - Qu_\theta}{-Kv + [Q\hat{u}_\theta - K\bar{u}_\theta - (q^2 - k^2)u_\theta]} \quad (3.9)$$

where the expected δ^2 term in the numerator disappears by virtue of the definitions (3.7,3.5). Then introducing $v = f/g$ we can linearise these equations and obtain

$$\tilde{\phi} = \Lambda \begin{pmatrix} (p^2 - k^2)\tilde{u}_\theta + K\tilde{\bar{u}}_\theta - Pu_\theta & 0 \\ -K & P\tilde{u}_\theta - K\bar{u}_\theta - (p^2 - k^2)u_\theta \end{pmatrix} \phi \quad (3.10)$$

and similarly for $\hat{\phi}$, where $\phi = (f, g)^T$. The factor Λ is determined by the condition that the shifts with \sim and $\hat{}$ must commute, from which it follows we should take $\Lambda \propto 1/U_\theta$, where (compare with (3.3))

$$(U_\theta)_{nm} = (a+b)A\tau^n \sigma^m - (a+b)B\tau^{-n} \sigma^{-m} + (a-b)C\dot{\tau}^n \dot{\sigma}^m - (a-b)D\dot{\tau}^{-n} \dot{\sigma}^{-m}. \quad (3.11)$$

Then with some algebra we obtain:

$$\phi_{n,m} = (p-k)^n (q-k)^m \begin{pmatrix} \rho_{n,m}(U_\theta)_{n,m}/(U_\theta)_{0,0} & 0 \\ \frac{K}{2k}[1 - \rho_{n,m}]/(U_\theta)_{0,0} & 1 \end{pmatrix} \phi_{0,0}, \quad (3.12)$$

where $\rho_{n,m}$ is the discrete “plane-wave factor”, defined by

$$\rho_{n,m} = \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m \rho_{0,0}. \quad (3.13)$$

From the result (3.12) we can reconstruct v of the 1SS:

$$v_{n,m} = \frac{2k(U_\theta)_{nm}\rho_{nm}v_{00}}{2k(U_\theta)_{00} + Kv_{00}(1 - \rho_{nm})}, \quad (3.14)$$

and finally $u_{nm}^{1SS} = \bar{u}_\theta + v$.

For a more explicit form showing the A, B, C, D dependence, we redefine the constant $\rho_{0,0}$ in (3.13) so that the denominator becomes proportional to $1 + \rho_{nm}$. Furthermore, after scaling A, B, C, D so that \bar{u}_θ of (3.7) becomes u^{0SS} of (3.3) we can write the 1SS (3.14) as

$$u_{nm}^{1SS} = \left[A\tau^n \sigma^m \left(1 + \rho_{nm} \frac{(a-k)(b-k)}{(a+k)(b+k)} \right) + B\tau^{-n} \sigma^{-m} \left(1 + \rho_{nm} \frac{(a+k)(b+k)}{(a-k)(b-k)} \right) \right. \\ \left. + C\dot{\tau}^n \dot{\sigma}^m \left(1 + \rho_{nm} \frac{(a-k)(b+k)}{(a+k)(b-k)} \right) + D\dot{\tau}^{-n} \dot{\sigma}^{-m} \left(1 + \rho_{nm} \frac{(a+k)(b-k)}{(a-k)(b+k)} \right) \right] / (1 + \rho_{nm}). \quad (3.15)$$

4. N -soliton solutions and Hirota bilinear form

We will now present the main result of the paper, which is a general N -soliton solution of Q3. This solution can be written in the following form:

$$u_{nm} = A\tau^n \sigma^m \frac{F(n, m, \alpha + 1, \beta + 1)}{F(n, m, \alpha, \beta)} + B\tau^{-n} \sigma^{-m} \frac{F(n, m, \alpha - 1, \beta - 1)}{F(n, m, \alpha, \beta)} \\ + C\dot{\tau}^n \dot{\sigma}^m \frac{F(n, m, \alpha + 1, \beta - 1)}{F(n, m, \alpha, \beta)} + D\dot{\tau}^{-n} \dot{\sigma}^{-m} \frac{F(n, m, \alpha - 1, \beta + 1)}{F(n, m, \alpha, \beta)}, \quad (4.1)$$

where the tau-function F is given by

$$F(n, m, \alpha, \beta) = \sum_{\mu_j \in \{0,1\}} \exp \left(\sum_{j=1}^N \mu_j \eta_j + \sum_{1 \leq i < j \leq N} \mu_i \mu_j a_{ij} \right), \quad (4.2)$$

where

$$\exp \eta_j \equiv \rho_{nm\alpha\beta}(k_j) := \left(\frac{p + k_j}{p - k_j} \right)^n \left(\frac{q + k_j}{q - k_j} \right)^m \left(\frac{a - k_j}{a + k_j} \right)^\alpha \left(\frac{b - k_j}{b + k_j} \right)^\beta \rho_0, \quad (4.3)$$

$$\exp a_{ij} \equiv A_{ij} := \left(\frac{k_i - k_j}{k_i + k_j} \right)^2, \quad (4.4)$$

and the coefficients in (4.1) are restricted by

$$AB(a + b)^2 - CD(a - b)^2 = -\frac{\delta^2}{16ab}. \quad (4.5)$$

One way in which this result can be derived is by using the Miura transformation given in the Appendix and using a number of relations following from the direct linearisation structure of the NQC equation of [3, 4]. We will not exhibit this approach here for lack of space. Rather, we will argue the result based on the connection with the Hirota-Miwa difference equations, [20, 21].

Having established the 1SS (3.15), which is special case of the form (4.1) with (4.2), we checked the 2SS suggested by Hirota's perturbative approach

$$F(n, m, \alpha, \beta) = 1 + \rho_{nm\alpha\beta}(k_1) + \rho_{nm\alpha\beta}(k_2) + A_{12}\rho_{nm\alpha\beta}(k_1)\rho_{nm\alpha\beta}(k_2). \quad (4.6)$$

and verified that the phase factor A_{ij} is given as in (4.4). We have also explicitly verified that the 3SS version solves Q3.

The form of the phase factor (4.4) allows us to identify the solution to be in the Hirota-Miwa hierarchy [20, 21], within the reduction $-q_i = p_i \equiv k_i$ (up to some notational conventions for the parameters), except that we now seem to have a 4D system n, m, α, β with diagonal $[(\alpha \pm 1, \beta \pm 1)]$ and anti-diagonal $[(\alpha \pm 1, \beta \mp 1)]$ reductions.

The N -soliton F in (4.2) solves the Hirota-Miwa bilinear difference equations in any 3 of the 4 indices (along with some higher equations in the hierarchy), e.g.,

$$(q + a)F_1F_{23} - (a + p)F_2F_{13} + (p - q)F_{12}F_3 = 0, \quad (4.7a)$$

$$(q + b)F_1F_{24} - (b + p)F_2F_{14} + (p - q)F_{12}F_4 = 0, \quad (4.7b)$$

$$(p - q)F_{123}F + (q - a)F_{13}F_2 - (p - a)F_1F_{23} = 0, \quad (4.7c)$$

$$(p - q)F_{124}F + (q - b)F_{14}F_2 - (p - b)F_1F_{24} = 0. \quad (4.7d)$$

Here we have used the notation where only the shifted index is indicated, e.g., $F_{n,m+1,\alpha+1,\beta+1} \equiv F_{234}$. The shift in the negative direction is indicated by a bar, e.g., $F_{n,m,\alpha-1,\beta+1} \equiv F_{\bar{3}4}$. In the following we also need the various shifts of these equations. (The equations above would be more symmetric if we were to change $a \rightarrow -a$ $b \rightarrow -b$, cf. (4.3).)

Using the above equations we can show that (4.1) solves (2.4). After substitution we collect terms with different factors of $\tau^n, \sigma^m, \dot{\tau}^n, \dot{\sigma}^m$ or equivalently different combinations of the coefficients A, B, C, D , except that AB, CD and δ^2 should go together. For example the coefficient of D^2 (or $\dot{\tau}^{-2n}\dot{\sigma}^{-2m}$) is quartic in the F 's, but it can be decomposed into various combinations of Hirota-Miwa equations (4.7) as follows

$$F_{2\bar{3}} \times (\text{coeff. of } D^2) \propto [(4.7c)_{\bar{3}}F_{12\bar{3}4} - (4.7d)_{\bar{3}}F_{12}][(a+q)F_2F_{\bar{3}4} - (b+q)FF_{2\bar{3}4}] \\ - [(4.7a)_{\bar{3}}F_{\bar{3}4} - (4.7b)_{\bar{3}}F][(a-p)F_2F_{12\bar{3}4} - (b-p)F_{12}F_{2\bar{3}4}].$$

and similarly for the coefficients of A^2, B^2, C^2 . The coefficient of AC , on the other hand, is proportional to

$$\{[-(4.7b)_3F + (4.7c)F_{34}][(a+p)F_2F_{12\bar{3}4} - (b+p)F_{12}F_{2\bar{3}4}] \\ + [(4.7b)_{\bar{3}4}F_{12} - (4.7a)F_{12\bar{3}4}][(a-q)F_2F_{34} + (b+q)FF_{234}]\}(b-p)(b-q) \\ - \{[(4.7d)_{\bar{3}4}F - (4.7c)F_{\bar{3}4}][(a+p)F_2F_{1234} + (b-p)F_{12}F_{234}] \\ - [(4.7d)_3F_{12} - (4.7a)F_{1234}][(a-q)F_2F_{\bar{3}4} - (b-q)FF_{2\bar{3}4}]\}(b+p)(b+q)$$

and similar formulae can be written for the coefficients of AD, BC, BD .

This leaves the equation containing AB, CD , and δ^2 , which can be written as

$$AB\mathcal{R}(b) + CD\mathcal{R}(-b) + \delta^2\mathcal{S} = 0 \quad (4.8)$$

where \mathcal{R} and \mathcal{S} are some expression in F and the parameters and \mathcal{S} does not depend on b . The reflection property with respect to b is obvious when one considers the role of b in (2.3, 3.2, 4.1, 4.3).

One of the fundamental assumptions before was that although the coefficients A, B, C, D may depend on δ , the soliton part F does not. Thus equation (4.8) should hold under (4.5), whether or not $\delta = 0$. If $\delta = 0$ we get immediately the condition

$$(a+b)^2\mathcal{R}(b) + (a-b)^2\mathcal{R}(-b) = 0. \quad (4.9)$$

and then using this we get for $\delta \neq 0$

$$\mathcal{R}(b) + 16ab(a-b)^2\mathcal{S} = 0. \quad (4.10)$$

If (4.10) holds, so does its $b \rightarrow -b$ reflection, and they together imply (4.9). In full

detail equation (4.10) reads

$$\begin{aligned}
& F_{12\bar{3}4}F_{13\bar{4}}F_2F(a+p)(a-p)(a-q)(b+p)(b-p)(b+q) \\
& - F_{12\bar{3}4}F_1F_{2\bar{3}4}F(a-p)(a+q)(a-q)(b+p)(b+q)(b-q) \\
& + F_{12\bar{3}4}F_1F_2F_{3\bar{4}}(a-p)(a-q)(b+p)(b+q)(p+q)(p-q) \\
& + F_{12\bar{3}4}F_{1\bar{3}4}F_2F(a+p)(a-p)(a+q)(b+p)(b-p)(b-q) \\
& - F_{12\bar{3}4}F_1F_{2\bar{3}4}F(a+p)(a+q)(a-q)(b-p)(b+q)(b-q) \\
& + F_{12\bar{3}4}F_1F_2F_{\bar{3}4}(a+p)(a+q)(b-p)(b-q)(p+q)(p-q) \\
& + F_{12}F_{1\bar{3}4}F_{2\bar{3}4}F(a-p)(a+q)(b+p)(b-q)(p+q)(p-q) \\
& - F_{12}F_{1\bar{3}4}F_2F_{3\bar{4}}(a-p)(a+q)(a-q)(b+p)(b+q)(b-q) \\
& + F_{12}F_{1\bar{3}4}F_{2\bar{3}4}F(a+p)(a-q)(b-p)(b+q)(p+q)(p-q) \\
& - F_{12}F_{1\bar{3}4}F_2F_{\bar{3}4}(a+p)(a+q)(a-q)(b-p)(b+q)(b-q) \\
& + F_{12}F_1F_{2\bar{3}4}F_{3\bar{4}}(a+p)(a-p)(a-q)(b+p)(b-p)(b+q) \\
& + F_{12}F_1F_{2\bar{3}4}F_{\bar{3}4}(a+p)(a-p)(a+q)(b+p)(b-p)(b-q) \\
& + 4F_{12}F_1F_2F(a-b)^2(p+q)(p-q)ab = 0,
\end{aligned}$$

and it must also be expressible in terms of the equations in the Hirota-Miwa hierarchy. (We have verified that this equation is solved by the 3SS of the type (4.2).)

5. Conclusions

In this paper we have constructed NSS to Q3. This was done through an associated equation, the NQC-equation, which contains extra parameters a, b . The correspondence from NQC to $(Q3)_0$ is 4-to-1, labeled by the different sign combinations of a, b , and this leads us to a four-term background solution, from which the 1SS and then the NSS can be constructed. These solutions contain the new parameters a, b in an essential manner. An important observation is the relationship with the Hirota-Miwa equation with four variables.

Another proof based on the known NSS of the NQC equation, as well as other details and properties will be given elsewhere [18].

Acknowledgments

One of the authors (JH) would like to thank JJC Nimmo for discussions. JA was supported by the UK Engineering and Physical Sciences Research Council.

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Appendix: Miura transformation between $(Q3)_0$ and $(Q3)_\delta$

In the Section 3 we constructed a 1SS (3.15) starting with the most general OSS (3.3) and using the CAC-cube, in which the vertical direction associated with the Bäcklund transformation. Here we again use the CAC-cube, but now the bottom layer is $(Q3)_\delta$, the top layer $(Q3)_0$, with the sides providing linear equations by which a solution of $(Q3)_0$ can be transformed into a solution of $(Q3)_\delta$. For example the well known NSS, corresponding to $A = 1, B = C = D = 0$, which satisfies (3.4) but not (3.5), can be

transformed into a solution of $(Q3)_\delta$ and therefore this transformation should generate at least a B term [18].

A.1. Derivation of Miura transformation

In order to get the $\delta = 0$ version of (2.6) on the top layer of the cube we use a simple scaling argument. Let us again denote the vertical shift by a bar and introduce the scaling $\bar{u} = v/\epsilon$, with $\epsilon \rightarrow 0$, where v solves $(Q3)_0$. In this limit the top equation immediately becomes (2.4). For the side equation we get first

$$P(uv + \tilde{u}\tilde{v}) - \epsilon R(u\tilde{u} + \frac{1}{2}v\tilde{v}) = (p^2 - r^2) \left(u\tilde{v} + \tilde{u}v + \frac{\epsilon\delta^2}{4PR} \right),$$

and a similar equation for the shift in the other lattice direction associated with q . Arranging the parameter R to be of order ϵ , i.e. $R = \epsilon R_0$ with R_0 finite and nonzero as $\epsilon \rightarrow 0$, yields in dominant order in ϵ the equation

$$P(uv + \tilde{u}\tilde{v}) - R_0 v\tilde{v} = (p^2 - r^2) \left(u\tilde{v} + \tilde{u}v + \frac{\delta^2}{4PR_0} \right).$$

However, we must respect the parametrization of (2.5), which now reads

$$\epsilon^2 R_0^2 = (r^2 - a^2)(r^2 - b^2),$$

and this can only be of the right order in ϵ if $r^2 = a^2 + \epsilon^2 r_0^2$, or $r^2 = b^2 - \epsilon^2 r_0^2$ with r_0 finite as $\epsilon \rightarrow 0$. Hence, $R_0^2 = r_0^2(a^2 - b^2)$, but since r_0 is arbitrary, we may keep R_0 arbitrary and only remember that either $r^2 = a^2$ or $r^2 = b^2$. Thus, we are led to the following Miura transformation between $(Q3)_\delta$ and $(Q3)_0$:

$$P(uv + \tilde{u}\tilde{v}) - Rv\tilde{v} = (p^2 - r^2) \left(\tilde{u}v + u\tilde{v} + \frac{\delta^2}{4PR} \right), \quad (\text{A.1a})$$

$$Q(uv + \hat{u}\hat{v}) - Rv\hat{v} = (q^2 - r^2) \left(\hat{u}v + u\hat{v} + \frac{\delta^2}{4QR} \right), \quad (\text{A.1b})$$

where we have suppressed the suffix 0.

In (A.1) v is the solution of $(Q3)_0$ and u a corresponding solution of $(Q3)_\delta$, and we note that the resulting equations are actually *linear* in the new solution u :

$$[Pv - (p^2 - a^2)\tilde{v}]u + [P\tilde{v} - (p^2 - a^2)v]\tilde{u} = Rv\tilde{v} + \frac{\delta^2(p^2 - a^2)}{4PR}, \quad (\text{A.2a})$$

$$[Qv - (q^2 - a^2)\hat{v}]u + [Q\hat{v} - (q^2 - a^2)v]\hat{u} = Rv\hat{v} + \frac{\delta^2(q^2 - a^2)}{4QR}, \quad (\text{A.2b})$$

where we have chosen $r^2 = a^2$. It can be shown that the Miura transformation (A.2) commute on the two-dimensional lattice, as a consequence of which we can “integrate” the linear relations (A.2a) and (A.2b) simultaneously in both lattice directions to obtain a well-defined solution, since the lattice equation obeyed by v is a member of the equations consistent around the cube.

Other Miura type transformations between various members of the ABS list of lattice equations, and related equations that can be obtained by degeneration have been constructed, and will be published separately [22].

A.2. From seed of (Q3)₀ to the seed of (Q3)_δ

We now apply the Miura transformation (A.1) starting with v given by the background solution $v_{n,m} = -\tau^n \sigma^m / (a+b)$. In that case the coefficients of the linear relation (A.2a) are given by:

$$Pv - (p^2 - a^2)\tilde{v} = \frac{P}{p-b}\tau^n \sigma^m, \quad P\tilde{v} - (p^2 - a^2)v = -\frac{P}{p+b}\tau^{n+1}\sigma^m, \quad (\text{A.3})$$

and similarly for the coefficients of (A.2b) interchanging the roles of p and q and of n and m . Inserting these coefficients we obtain the linear equations

$$u - \frac{p-b}{p+b}\tau\tilde{u} = \frac{R(p-b)}{P(a+b)^2}\tau^{n+1}\sigma^m + \frac{\delta^2}{4(p+b)R}\tau^{-n}\sigma^{-m}, \quad (\text{A.4a})$$

$$u - \frac{q-b}{q+b}\sigma\hat{u} = \frac{R(q-b)}{Q(a+b)^2}\tau^n\sigma^{m+1} + \frac{\delta^2}{4(q+b)R}\tau^{-n}\sigma^{-m}. \quad (\text{A.4b})$$

The first relation can be integrated w.r.t. the variable n , using $\dot{\tau}^n$ defined in (3.2) as integrating factor, and we obtain

$$u_{0,m} - \dot{\tau}^n u_{n,m} = \frac{R\sigma^m}{2a(a+b)^2} \left[\left(\frac{p+a}{p-a} \right)^n - 1 \right] - \frac{\delta^2 \sigma^{-m}}{8bR} \left[\left(\frac{p-b}{p+b} \right)^n - 1 \right],$$

What is significant here is that the coefficients on the right hand side no longer depend on the lattice parameter p , and that p and q only occur through the discrete “exponentials”. Obviously a similar relation can be obtained from (A.4b) integrating w.r.t. variable m , and the result is given by simply interchanging p and q and the roles of n and m ;

$$u_{n,0} - \dot{\sigma}^m u_{n,m} = \frac{R\tau^n}{2a(a+b)^2} \left[\left(\frac{q+a}{q-a} \right)^m - 1 \right] - \frac{\delta^2 \tau^{-n}}{8bR} \left[\left(\frac{q-b}{q+b} \right)^m - 1 \right],$$

with $\dot{\sigma}$ defined in (2.3).

Now we can either take the first result and add a multiple of the second result at $n = 0$, or take the second result and add a multiple of the first result at $m = 0$, thus eliminating the intermediate terms $u_{0,m}$ and $u_{n,0}$ respectively. In either way we get the following result:

$$u_{nm} = -\tau^n \sigma^m \frac{R}{2a(a+b)^2} + \tau^{-n} \sigma^{-m} \frac{\delta^2}{8bR} + \dot{\tau}^{-n} \dot{\sigma}^{-m} \left(u_{00} - \frac{\delta^2}{8bR} + \frac{R}{2a(a+b)^2} \right).$$

Thus, starting with the background solution for (Q3)₀ with $A = 1, B = C = D = 0$ we generated a solution of (Q3)_δ with $A = -R/(2a(a+b)^2)$, $B = \delta^2/(8bR)$, $C = 0$, $D \neq 0$, which indeed satisfies the constraint (3.5).

This three-term solution can be suggestively written in as follows:

$$u_{n,m} = A\tau^n \sigma^m + C\dot{\tau}^{-n} \dot{\sigma}^{-m} + D\tau^{-n} \sigma^{-m},$$

which contains all but one of the possible sign interchangements of the parameters a and b in the discrete exponential factors. This seems to suggest that there is one term missing, which could have been obtained by using a Miura transformation with $r = \pm b$, instead of $r = \pm a$ as we have done. Thus we reach the ansatz (3.3).